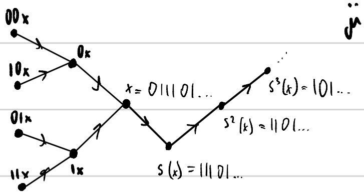
Ergodic Theory and Measured Group Theory
Lecture 2

Examples of pmp transformations (continued).

- One-sided shift on $(2^{\mathbb{N}}, \underset{\mu}{\underset{\shortparallel}{\nu^{\mathbb{N}}}})$, i.e. $s : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$, $(x_n) \mapsto (x_{n+1})$, where ν is a prob. measure on $2 := \{0,1\}$.



Every basic open set is of the form $U_w := \{x \in 2^{\mathbb{N}} : w \subseteq x\}$ for some $w \in 2^{<\mathbb{N}}$. These generate the Borel σ-algebra and s^{-1} preserves complements and unions, it's enough to show that $\mu(s^{-1}(U_w)) = \mu(U_w)$. But $s^{-1}(U_w) = \{x \in 2^{\mathbb{N}} : x = *\underbrace{\;w\;}**\dots\}$ so $\mu(s^{-1}(U_w)) = \nu(w_0) \cdot \nu(w_1) \cdot \ldots \cdot \nu(w_{n-1}) = \mu(U_w)$, $n := |w|$.

Obs. Baker's map is isomorphic to the one-sided shift on $(2^{\mathbb{N}}, \nu^{\mathbb{N}})$ with $\nu := \{\frac{1}{2}, \frac{1}{2}\}$.

Proof. The baker's map b maps the binary rep. $x = 0.x_0x_1x_2\dots$ to $0.x_1x_2x_3\dots$, so mapping $x \overset{\varphi}{\mapsto} (x_0, x_1, x_2, \dots)$ maps $[0,1)$ to $2^{\mathbb{N}}$ and pushes the Lebesgue measure forward to $\{\frac{1}{2}, \frac{1}{2}\}^{\mathbb{N}}$. This map is equivariant:
$$\varphi \circ b = s \circ \varphi.$$
□

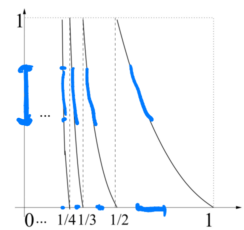
o Two-sided shift on $(2^{\mathbb{Z}}, \nu^{\mathbb{Z}})$. $s: 2^{\mathbb{Z}} \to 2^{\mathbb{Z}}$, $(x_n)_{n\in\mathbb{Z}} \mapsto (x_{n+1})_{n\in\mathbb{Z}}$.

This is a 1-1 map and clearly pmp because, if $U_w := \{x \in 2^{\mathbb{Z}} : x_{-k}x_{-k+1}\cdots x_n = w\}$ then $s^{-1}(U_w) = \{x \in 2^{\mathbb{Z}} : x_{-k-1}x_{-k}\cdots x_{n-1} = w\}$, so $\mu(s^{-1}(U_w)) = \mu(U_w)$.

o Gauss map. $g : [0,1) \to [0,1)$

$$x \mapsto \begin{cases} 0 & \text{if } x = 0 \\ \frac{1}{x} - [\frac{1}{x}] & \text{o.w.} \end{cases}$$

$x := [a_0, a_1, a_2, a_3, \ldots] = \dfrac{1}{a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \ldots}}}}$. then $g(x) = [a_1, a_2, a_3, \ldots]$.



$g|_{(\frac{1}{n+1}, \frac{1}{n}]}(x) = \frac{1}{x} - n$. This g doesn't preserve the Lebesgue measure, but it preserves the measure $d\mu(x) := \frac{\log 2}{1+x} d\lambda(x)$, i.e. for $A \subseteq [0,1)$, $\mu(A) = \int_A \frac{1}{\log 2 (1+x)} d\lambda(x)$, so $\mu([0,1)) = 1$.

Claim. g preserves μ.

Proof. Because the intervals $A := [0, a)$ generate the Borel σ-alg, it's enough to show $\mu(g^{-1}(A)) = \mu(A)$.

$$g^{-1}([0,a)) = \bigsqcup_{n=1}^{\infty} \left(\frac{1}{n+a}, \frac{1}{n}\right], \text{ so } \mu(g^{-1}(A)) = \sum_{n=1}^{\infty} \mu\left(\left(\frac{1}{n+a}, \frac{1}{n}\right]\right)$$

$$= \sum_{n=1}^{\infty} \int_{\frac{1}{n+a}}^{\frac{1}{n}} \frac{\log 2}{1+x} d\lambda(x) = \frac{1}{\log 2} \sum_{n=1}^{\infty} \log\left(1 + \frac{1}{n}\right) - \log\left(1 + \frac{1}{n+a}\right) =$$

$$= \frac{1}{\log 2}\left(\sum_{n=1}^{\infty} \log(n+1) - \log n - \log(n+a+1) + \log(n+a)\right)$$

$$= \frac{1}{\log 2} \lim_{N\to\infty} \log(N+1) + \log(1+a) - \log(N+a+1)$$

$$= \frac{1}{\log 2}\left(\log(1+a) + \lim_{N\to\infty} \log\frac{N+1}{N+a+1}\right)$$

$$= \frac{1}{\log 2} \log(1+a) = \frac{1}{\log 2}\int_0^a \frac{1}{1+x}\, d\lambda(x) = \mu([0,a)). \quad \square$$

<u>Lemma (Change of variable).</u> $T:(X,\mu)\to(X,\mu)$ is μ-preserving iff $\forall f\in L^1(X,\mu)$, $\int f(x)\,d\mu(x) = \int f(Tx)\,d\mu(x)$.

Proof. $\Leftarrow$. Trivial: $f := \mathbb{1}_A$ for $A\subseteq X$, so $\int \mathbb{1}_A\, d\mu = \int \mathbb{1}_A(Tx)\,d\mu(x)$

$= \int \mathbb{1}_{T^{-1}(A)}\,d\mu = \mu(T^{-1}(A))$.

$\Rightarrow$. We know that $\mu(T^{-1}(A)) = \mu(A)$ for every measurable set $A\subseteq X$, so $\int \mathbb{1}_A(x)\,d\mu(x) = \int \mathbb{1}_A(Tx)\,d\mu(x)$.

Thus, we know this for linear comb. of indicator functions, called simple functions. For any $f\in L^1(X,\mu)$ $f\geq 0$ $\exists$ sequence (f_n) of simple functions s.t. $f_n \nearrow f$.

Thus, the Monotone Convergence theorem gives

$\int f\,d\mu = \int f(Tx)\,d\mu(x)$. Then for any $f\in L^1$, $f = f^+ - f^-$. $\square$

Koopman operator. To any pmp $T:(X,\mu)\to(X,\mu)$ we associate an operator $\hat{T}: L^1(X,\mu)\to L^1(X,\mu)$, called **Koopman**.

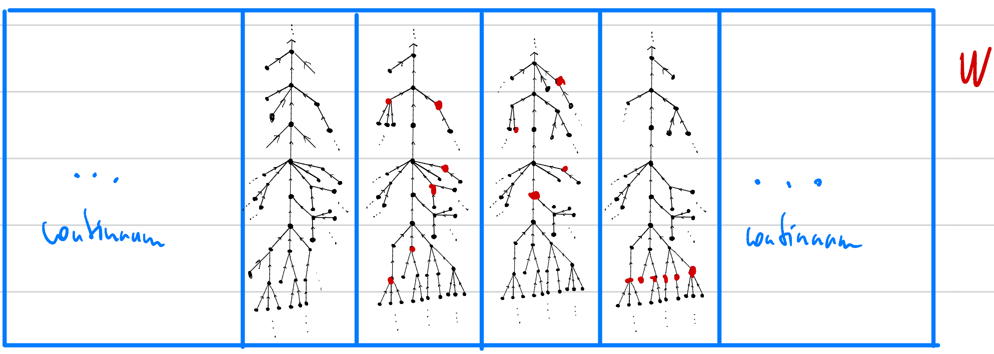
$$f \mapsto f\circ T$$

By the change of var formula, $\|\hat{T}f\|_1 = \int |f(Tx)|\,d\mu(x) = \int |f|\,d\mu = \|f\|_1$, so $\hat{T}$ is a linear isometry (not necessarily surjective).

We abuse notation and write T for $\hat{T}$, so $Tf := f\circ T$.

Wandering sets and recurrence. Let T be a pmp trans. on (X,μ).

Def. A meas. set $W\subseteq X$ is called **wandering** if $W, T^{-1}(W), T^{-2}(W), \ldots$ are pairwise disjoint.



Obs. Because T is pmp, wandering sets are null.

Proof. The sets $W, T^{-1}(W), T^{-2}(W), \ldots$ are pairwise disjoint and have equal measure, and hence $\mu(X) < \infty$, it must be 0. $\square$

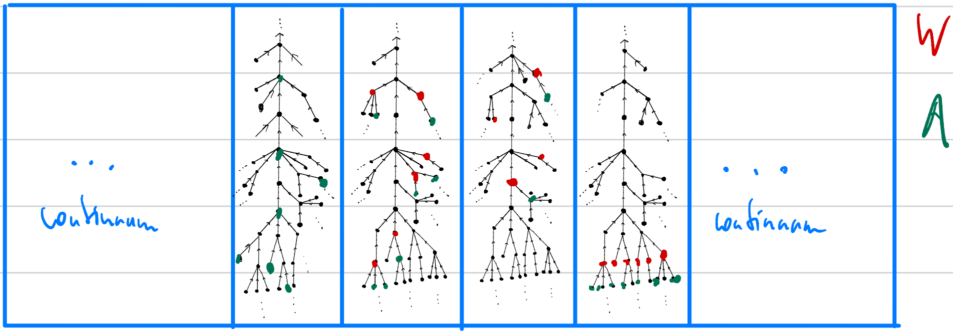
Def. Call a set $A \subseteq X$ *T-forward recurrent* if $\forall x \in A$, $\exists n \geq 1$ s.t. $T^n(x) \in A$.

Notation. For measurable sets $A, B \subseteq X$, write $A =_\mu B$ if $A \Delta B$ is null.

(Poincaré recurrence)

Cor. For every meas. $A \subseteq X$ is T-forward recurrent a.e. (assuming T is pmp). More precisely, $\exists$ meas. $A' \subseteq X$ s.t. $A =_\mu A'$, A' is T-forward-recurrent.

Proof. Let $W := \{x \in A : \forall n \geq 1 \ T^n(x) \notin A\}$, meas. since T is meas. Note that $T^{-n} W \cap W = \emptyset$ $\forall n \geq 1$. This implies $T^{-n} W \cap T^{-m} W = \emptyset$ $\forall n \neq m$ so W is wandering, hence null. Thus, $\bigcup_{n=0}^{\infty} T^{-n} W$ is null. Let $A' := A \setminus \left(\bigcup_{n=0}^{\infty} T^{-n} W\right)$. $\square$



Invariant sets and functions. For T on (X,μ), recall the eq. rel. E_T.

Def. For an equivalence relation E on the set X, call $A \subseteq X$ *E-invariant* if A is a union of E-classes.
Call a function $f : X \to Y$ *E-invariant* if it is constant on every E-class.

We call a set/function on (X,μ) *T-invariant* if it is E_T-invariant.

Obs. (a) A set $A \subseteq X$ is T-inv. $\iff$ $T^{-1}A = A$.
(b) A function $f : X \to Y$ is T-inv. $\iff$ $Tf = f$.

Def. For an eq. rel. E on X and $A \subseteq X$, call
$$[A]_E := \{x \in X : \exists y \in A \;\; x E y\}$$
the *E-saturation* of A.

Obs. For a set $A \subseteq X$, $[A]_{E_T} = \bigcup_{n,m \in \mathbb{Z}} T^{-n}(T^m A)$.

Cor (of recurrence). If T is pmp, then for every meas. $A \subseteq X$,

$$[A')_{E_T} = \bigcup_{n=0}^{\infty} T^{-n} A', \quad \text{for some } A' =_\mu A.$$

Proof. $A =_\mu A'$, A' recurrent, so $[A']_{E_T} = \bigcup_{n=0}^{\infty} T^{-n} A'$. $\square$